Modeling time and spatial dependence of degradation processes through gamma processes

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Why spatial variability and uncertainty ?

The degradation in the structure varies in space and in time
Qualifies the non-homogeneity in the mechanical and physical properties.
Uncertainty : the intrinsic aleatory of the material

Source of spatial variability

- intrinsic aleatory of the composition
- the environmental conditions during the life of the structure (temperature, humidity, chloride penetration)
- the implementation conditions (workmanship : anisotropy and heterogeneity)

➥ It influences the predictions of degradation and the reliability of the structural component

Probabilistic approaches for modeling degradation process

1. Physical model approach
    - Simulation of physical quantities varying in space and in time.
    - Transport equation with random coefficient and data (Stewart et all ; Shinozuka et all,. . .)
    - ➥ Curse of dimensionality
    - ➥ Huge cost of the resolution (MC, Galerkin, Sparse grid)
2. Meta-model approach (classical model)
    - Based on measurable quantities to fit the model
    - Stochastic process (Gamma process, Brownian motion,. . .)

Extension of the classical model

- Incorporate the spatial variability
- Predict the degradation evolution temporally and Spatially
- ➤ based on the classical temporal model (gamma process)
- ➤ Mathematical and numerical computation benefits
- ➥ Spatio-temporal random field

Gamma process

(X_t) is a Gamma process with parameter $(\alpha, \beta) > 0$ (α increasing function) if

- $X_0 = 0$ p.s
- X_t has independent positive increments
- $X_{t+s} - X_s \sim gamma(\alpha(t+s) - \alpha(s), \beta)$ (gamma distribution)

- $\alpha(\cdot)$ linear : X_t is stationary process
- $\alpha(\cdot)$ non-linear : X_t is non-stationary process

properties

- The mean $\mu(t) = \frac{\alpha(t)}{\beta}$ and variance $\sigma = \frac{\alpha(t)}{\beta^2}$
- Scaling $\gamma X_t = Gamma(\alpha, \beta/\gamma)$
- Additivity $Gamma(\alpha_1, \beta) + Gamma(\alpha_2, \beta) = Gamma(\alpha_1 + \alpha_2, \beta)$
- $\mathbb{E}[\log(X_t)] = \psi(\alpha(t)) - \log(\beta)$
- $Var[\log(X_t)] = \psi_1(\alpha(t))$

where $\psi(r) = \frac{\Gamma'(r)}{\Gamma(r)}$ digamma function and ψ_1 its derivative.

Spatio-temporal random field

A sequence of random variables which is indexed in spatio-temporal box

$$\{G(t,x);\ (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d\}.$$

1. Separable random field
   - No interaction between time and space
   - Widely used even in situations in which they are not always physically justifiable

two constructions

- additive field $G(t,x) = X(t) + Y(x)$
- multiplicative field $G(t,x) = X(t)Y(x)$

2. Everything else are non-separable random field

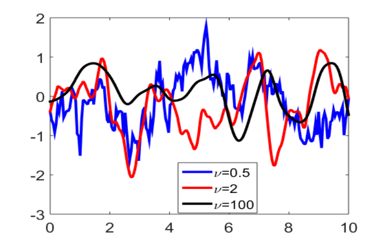
Stationary gaussian random field

$Y(x)$ is defined by the covariance of Matèrn model :

$$c(r) = \frac{\sigma^2 2^{\nu-1}}{\Gamma(\nu)} \left(\frac{\sqrt{2\nu} r}{lc} \right)^{\nu} \mathcal{K}_{\nu} \left(\frac{\sqrt{2\nu} r}{lc} \right)$$

- exponential $(\nu = 0.5)$, $c(r) = \sigma^2 e^{-\frac{r}{lc}}$
- Gaussian $(\nu = \infty)$, $c(r) = \sigma^2 e^{-\frac{r^2}{2lc^2}}$

σ^2 variance, lc correlation length, $\mathcal{K}_{\nu}$ modified Bessel function, $\nu > 0$ smoothness parameter.



Degradation model

$X_t = Ga(\alpha(t), \eta)$ Gamma process, $Z(x) = e^{Y(x)}$ log-normal field,

$$G_t(x) := X_t Z(x) \sim Ga(\alpha(t), \eta e^{-Y(x)})$$

Some raison for choosing log-normal distribution

- Products of positive independent random variables converges to a log-normal (CLM)
- Mathematical and statistical benefits in practice (ML, covariance)
- $Z \sim Z^{-1}$ thus $G_t(x) \sim X_t / Z(x)$ in law.

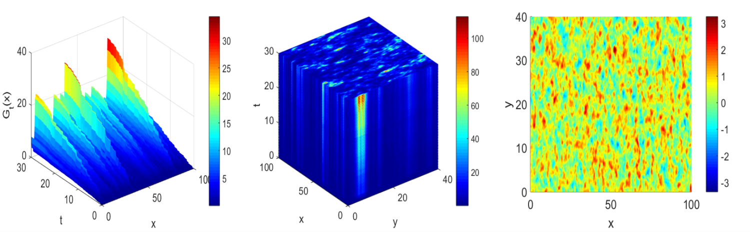


Figure – Stationary model,(left 1D, 2D middle). Gaussian field (right)

Methodology and approach

Inference

- Spatial parameters :
  σ^2 variance of Y
  l_c correlation length
  ν smoothness parameter
- Temporal parameters :
  $\alpha(t)$ shape parameter.
  $\eta = e^{\mu}$ deterministic scale.

Quantities of interest

- marginal law de G_t and failure time

$$T = \inf_{t>0} \Big(G_t(\cdot) \geq g_F \Big).$$

- residual life time

$$RL_t(\tau) = \Big(\tau > 0;\ G_{t+\tau} \geq g_F \big| G_t = g_t \Big)$$

Method of inference

- Method of moments
- maximum likelihood method

Methodology

- Estimation using realisations of G_t
- Estimation by quadrature

Marginal density of $G_t(x)$

$$f_t(v) = \frac{\eta^{\alpha(t)} v^{(\alpha(t)-1)}}{\Gamma(\alpha(t))} \int_{\mathbb{R}} \exp\Big(-v\eta \exp(\sigma y) + \alpha(t)\sigma y\Big)\xi(y)dy.$$

where ξ is the density of gaussian random variable $N(0,1)$

$$f_t^m(z) := \frac{\eta^{\alpha(t)} z^{(\alpha(t)-1)}}{\Gamma(\alpha(t))} \sum_{j=1}^{m} \exp\Big(-z\eta \exp(\sigma y_j) + \alpha(t)\sigma y_j\Big).$$

$\{y_j\}_{j=1}^m$ m Hermite's polynomial roots, $\{w_j\}_{j=1}^m$ their weights.
The order m is given by the stopped criterion,

$$|f_t^m - f_t^{m-1}| \leq \epsilon,$$

where $\epsilon > 0$ is a convenient threshold

failure distribution

- failure time distribution

$$F_T(t) \approx 1 - \int_0^{g_F} f_t^m(z)dz,$$

$$\tilde{F}_T(t_i) = (MN_x)^{-1} \sum_{j,k}^{N_x,M} \mathbb{I}_{\{G_{t_i}^k(x_j) \geq g_F\}}.$$

where $(G_{t_i}^k(x_j))$, $i = 1, \ldots, N_t$, $j = 1, \ldots, N_x$ realizations of G on the spatio-temporal box.

- Residual life distribution

$$P(RL_t > \tau) = \int_0^{g_F - g_t} \frac{u^{\delta_\tau \alpha - 1} g_t^{\alpha(t)-1}}{B(\delta_\tau \alpha, \alpha(t))(u + g_t)^{(\alpha(\tau+t)-1)}} \frac{f_{\tau+t}^m(u + g_t)}{f_t^m(g_t)} du$$

Inference Result (stationary case $\alpha(t) = \alpha t$)

Method of moments

- $\hat{\Upsilon}_Y$ is the experimental variogram of Y :

$$\min_{\sigma, lc, \nu > 0} \sum_{l=1}^{N_x} \left(\hat{\Upsilon}_Y(lh) + \sigma^2 cov_\nu(lh) - \sigma^2 \right)^2,$$

- $\hat{m}_1$, $\hat{m}_2$ are estimate of two first moments of $\log(\delta G_t)$, then :

$$\begin{pmatrix} \alpha \\ \eta \end{pmatrix} = \varphi^{-1} \begin{pmatrix} \hat{m}_1 \\ \hat{m}_2 \end{pmatrix}, \quad \text{où} \quad \varphi \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \psi(u\tau) - \log(v) \\ \psi_1(u\tau)) + \sigma^2 \end{pmatrix}$$

Maximum pseudo-likelihood method

- α et β maximize the pseudo-likelihood on the spatial grid

$$L(z) = \sum_{k=1}^{M} \sum_{j=1}^{N_t} \log \left(f_\tau^m(z_j^k) \right)$$

$N_x = 40,\ N_y = 20$		
M	σ^2	l_c
1	0.638(0.038)	0.913(0.087)
10	0.583(0.017)	1.06(0.06)

$N_x = 40, N_y = 20,\ N_t = 30$		
M	α	η
1	1.613(0.613)	2.660(0.71)
10	0.934(0.06)	1.833(0.134)

$N_x = 100, N_y = 40$		
M	σ^2	l_c
1	0.58(0.02)	0.949(0.051)
10	0.606(0.006)	1.007(0.007)

$N_x = 100, N_y = 40,\ N_t = 30$		
M	α	η
1	0.801(0.198)	2.184(0.236)
10	1.072(0.07)	2.123(0.175)

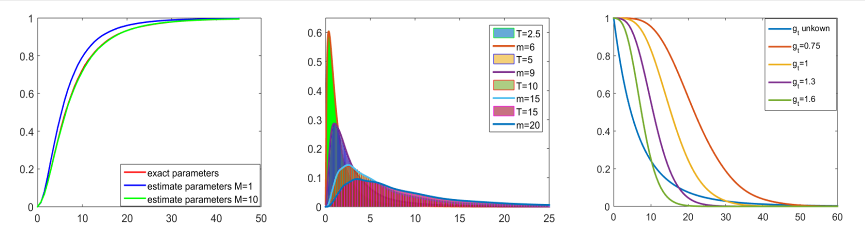


Figure – Failure distribution, marginal density, reliability and residual life

Conclusion

- New degradation model which incorporate both hazards temporal and spatial
- Statistical analysis for assessing the law of the model
- Estimation of the quantities of interest for reliability analysis

Perspective

- Extension to bivariate and non-stationary model.
- Scale parameter as spatio-temporal field derived from mathématical models (EDPS, EDS)
- Perturbation with Brownian motion to integrate measurement error
- Optimization approach on NDT measurements